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# A note on isomorphism theorems of Krasner $(m, n)$-hyperrings

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**Abstract** Recently, Krasner $(m, n)$-hyperrings were introduced and analyzed by Davvaz et. al. This is a suitable generalization of Krasner hyperrings. In this research work, we consider that if $I$ is a normal hyperideal of a Krasner $(m, n)$-hyperring $R$, then the quotient hyperring $[R : I^*]$ is an $(m, n)$-ring. Moreover, we prove that if $R$ is a multiplicative $(m, n)$-ary hyperring and $I$ is a normal hyperideal of $R$, then $[R : I^*]$ is an $(m, n)$-ring.

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**الملخص**

مؤخرًا، قدم كراسنر فوق الحلقات-$(m, n)$ وقام دافاس وآخرون بتحليلها. نقدم هنا تعميما مناسبا لفوق حلقات كراسنر. نثبت في هذا البحث أنه إذا كان $I$ فوق مثالي ناظميًا لفوق حلقة-$(m, n)$ فإن فوق حلقة القسمة $[R: I^*]$ حلقة-$(m, n)$. بالإضافة إلى ذلك، نثبت أنه إذا كانت $R$ فوق حلقة-$(m, n)$ ضربيّة وكان $I$ فوق مثالي ناظميًا فإن $[R: I^*]$ حلقة-$(m, n)$.

## 1 Introduction

Hypergroups were introduced in 1934 by a French mathematician Marty [14] at the Congress of Scandinavian Mathematicians. Since then, hundreds of papers and several books have been written on this topic. Nowadays, hyperstructures have a lot of applications in several domains of mathematics and computer science [1,5,13, 18,20,21].

The concept of $n$-ary groups were introduced about 80 years ago by Dörnte [8]. We can see the basic results on $n$-ary groups in [17]. Since then, many papers have been written on this topic [3,4,9,10,12]. In [7], Davvaz and Vougiouklis introduced the concept of $n$-ary hypergroups as a generalization of hypergroups in the sense of Marty which is a suitable generalization of $n$-ary groups. In [15], Davvaz et al. defined Krasner $(m, n)$-hyperrings as a generalization of $(m, n)$-rings and obtained several properties of Krasner $(m, n)$-hyperrings. Also, the isomorphism theorems of ring theory and Krasner hyperring theory are derived in the context of Krasner $(m, n)$-hyperrings.

The fundamental relation was introduced on hypergroups by Koskas [11] and then studied by Corsini [2]. It was introduced by Vougiouklis at the fourth AHA congress [19] and studied by many authors, for example see [6,19]. The fundamental relation on a hyperring was defined as the smallest equivalence relation so that the

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quotient would be the (fundamental) ring. Note that the commutativity with respect to both sum and product in the fundamental ring is not assumed.

In this paper, we define the fundamental relation $\eta^*$ on $R$ as the smallest equivalence relation on $R$ such that the quotient $[R : \eta^*]$ is an $(m, n)$-ring. Moreover, we observe that if $I$ is a normal hyperideal of Krasner $(m, n)$-hyperring, then $[R : I^*]$ is an $(m, n)$-ring and hence the quotient hyperrings considered in the isomorphism theorems are $(m, n)$-rings. Finally, we prove that if $R$ is a multiplicative $(m, n)$-ary hyperring and $I$ is a normal hyperideal of $R$, then $[R : I^*]$ is an $(m, n)$-ring.

## 2 Regular and strong regular relations

Let $R$ be a non-empty set and $n \in \mathbb{N}$, $n \geq 2$ and $f : R^n \longrightarrow \mathcal{P}^*(R)$, where $\mathcal{P}^*(R)$ is the set of all non-empty subsets of $R$. Then, $f$ is called an *n-ary hyperoperation* on $R$ and the pair $(R, f)$ is called an *n-ary hypergroupoid*. If $R_1, \ldots, R_n$ are non-empty subsets of $R$, then we define

$$f(R_1, R_2, \ldots, R_n) = \bigcup \{f(x_1, x_2, \ldots, x_n) : x_i \in R_i,\ i \in 1, 2, \ldots, n\}.$$

The sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by $x_i^j$. For $j < i$, $x_i^j$ is the empty set. An $n$-ary hypergroupoid $(R, f)$ will be called an *n-ary semihypergroup* if we have:

$$f\left(\overset{(i-1)}{x_1}, f\left(\overset{(n+i-1)}{x_i}\right), \overset{(2n-1)}{x_{n+i}}\right) = f\left(\overset{(j-1)}{x_1}, f\left(\overset{(n+j-1)}{x_j}\right), \overset{(2n-1)}{x_{n+j}}\right),$$

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots x_{2n-1} \in R$. Let the equation

$$y \in f\left(\overset{(i-1)}{x_1}, z_i, \overset{n}{x_{i+1}}\right),$$

has a solution $z_i \in R$ for every $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y \in R$. Then, $R$ is called *n-ary hypergroup*. An $n$-ary hypergroupoid $(R, f)$ is *commutative* if for all $\sigma \in S_n$, $f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$. A commutative $n$-ary hypergroupoid $(R, f)$ is called *canonical n-ary hypergroup* if the following axioms hold for all $1 \leq i, j \leq n$ and $x, x_i \in R$:

(i) There exists a unique element $0 \in R$ such that $x = f\left(\overset{(i-1)}{0}, x, \overset{(n-i)}{0}\right)$,

(ii) There exists a unique operation $-$ on $R$ such that $x \in f(\overset{n}{x_1})$ implies that $x_i \in f(-x_{i-1}, -x_{i-2}, \ldots -x_1, x, -x_n, \ldots, -x_{i+1})$.

*Example 2.1* Let $(G, \cdot)$ be a group and $H$ be a subgroup of $G$. We denote $[G : H] = \{xH : x \in G\}$, then $([G : H], f)$ is an $m$-ary hypergroup, where for all $x_i H \in [G : H]$, we have

$$F(\underbrace{x_1 H, x_2 H, \ldots, x_m H}_{m}) = \{zH : z \in x_1 H x_2 \ldots x_{m-1} H x_m\}.$$

*Example 2.2* Let $\{A\}_{g \in G}$ be a collection of non-empty sets where $(G, +)$ is a semigroup. Then, $S = \bigcup_{g \in G} A_g$ is an $m$-ary semihypergroup with respect to the following hyperoperation:

$$f(x_1, x_2, \ldots, x_m) = A_x,$$

where $x_i \in A_{g_i}$ and $x = \sum_{i=1}^m x_i$.

*Example 2.3* Let $R$ be a ring and $I$ an ideal of $R$. We define

$$f(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n + I$$

for every $x_1, \ldots, x_n$. Then, $(R, f)$ is an $n$-ary semihypergroup.

**Definition 2.4** [15] A Krasner $(m, n)$-hyperring is an algebraic structure $(R, f, g)$ which satisfies the following conditions:

(i) $(R, f)$ is a canonical $m$-ary hypergroup,

(ii) $(R, g)$ is an $n$-ary semigroup,
(iii) the $n$-ary operation is distributive with respect to the $m$-ary hyperoperation f, i.e., for every $x_i^{i-1}, x_{i+1}^n, x_i^m$, $1 \leq i \leq n$

$$g\left(\overset{(i-1)}{x_i}, f\left(\overset{m}{a_1}\right), \overset{n}{x_{i+1}}\right) = f\left(g\left(\overset{(i-1)}{x_1}, a_1, x_{i+1}^n\right), \ldots, g\left(\overset{(i-1)}{x_i}, a_m, \overset{n}{x_{i+1}}\right)\right).$$

(iv) 0 is a zero element (absorbing element) of the $n$-ary operation $g$, i.e., for every $x_2^n \in R$ we have

$$g(0, x_2^m) = g(x_2, 0, x_3^n) = \cdots = g(0, x_2^n) = 0.$$

**Definition 2.5** [16] If $(R, f, g)$ is an algebraic structure, such that

(i) $(R, f)$ is an m-ary hypergroup,
(ii) $(R, g)$ is an n-ary semihypergroup,
(iii) the $n$-ary operation is distributive with respect to the $m$-ary hyperoperation f, i.e., for every $x_i^{i-1}, x_{i+1}^n, x_i^m$, $1 \leq i \leq n$

$$g\left(\overset{(i-1)}{x_i}, f\left(\overset{m}{a_1}\right), \overset{n}{x_{i+1}}\right) = f\left(g\left(\overset{(i-1)}{x_1}, a_1, x_{i+1}^n\right), \ldots, g\left(\overset{(i-1)}{x_i}, a_m, \overset{n}{x_{i+1}}\right)\right),$$

then $(R, f, g)$ is called an $(m, n)$-ary hyperring.

In an $(m, n)$-ary hyperring if the hyper $m$-ary operation $f$ is an $m$-ary operation, then it is called as *multiplicative $(m, n)$-ary hyperring*.

Let $(R_1, f_1, g_1)$ and $(R_2, f_2, g_2)$ be two Krasner $(m, n)$-hyperrings. A mapping $\varphi : R_1 \longrightarrow R_2$ is called a *homomorphism* if for all $x_1^m \in R$ and $y_1^n$

$$\varphi(f_1(x_1^m)) = f_2(\varphi(x_1^m)), \quad \varphi(g_1(y_1^n)) = g_2(\varphi(y_1^n)).$$

*Example 2.6* Let $(R, f, g)$ be an $(m, n)$-ring and $I$ be an ideal of $R$. We define the following hyperoperation on $R$: For all $x_1^m \in R$, $\overline{g}(x_1^m) = f(g(x_1^m), \underbrace{I, I, \ldots, I}_{m-1})$. Then, $(R, f, \overline{g})$ is a multiplicative $(m, n)$-ary hyperring.

Let $(R, f, g)$ be a Krasner $(m, n)$-hyperring and $\rho$ be an equivalence relation on $R$. If $A$ and $B$ are non-empty subsets of $R$, then $A\overline{\rho}B$ means that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $b_1 \in B$, there exists $a_1 \in A$ such that $a_1 \rho b_1$ and $A\overline{\overline{\rho}}B$ means that for every $a \in A$ and $b \in B$, we have $a\rho b$.

**Definition 2.7** The equivalence relation $\rho$ on an $m$-ary hypergroup $(R, f)$ is called *regular* if for all $x_2, x_3, \ldots x_m \in R$, from $a\rho b$, it follows that

$$f(a, x_2, \ldots, x_m)\overline{\rho} f(b, x_2, \ldots, x_m),$$

and is called *strong regular* if for $x_2, \ldots x_m \in R$, $a\rho b$ implies that

$$f(a, x_2, \ldots, x_m)\overline{\overline{\rho}} f(a, x_2, \ldots, x_m).$$

**Proposition 2.8** *Let $R$ be a Krasner (m,n)-hyperring and $\rho$ be an equivalence relation on $R$. Then, $[R : \rho]$ is an $(m, n)$-ring if and only if $\rho$ is a strong regular relation.*

*Proof* It is straightforward. □

Let $(R, f)$ be an $m$-ary semihypergroup. We introduce a relation $\eta^*$ on $R$ as follows:
suppose that

$$A_1^0 = \{\{x\} : x \in R\}, \quad A_1^m[x_{1i}] = f(x_{11}^m), \ldots, A_1^{km}[x_{ki}] = f(A_1^{(k-1)m}[x_{ki}], x_{k2}^m).$$

Let $k \geq 0$ be an integer number. We say that

$$x\eta_k y \text{ if there exist } x_{i1}, x_{i2}, \ldots, x_{im} \text{ such that } \{x, y\} \subseteq A_1^{km}[x_{ki}].$$

Let $\eta = \bigcup_{n \geq 1} \eta_n$. Clearly, the relation $\eta$ is reflexive and symmetric. We denote the transitive closure of $\eta$ by $\eta^*$. We shall prove that the relation $\eta$ is transitive.

**Theorem 2.9** *Let $(R, f)$ be an $m$-ary semihypergroup. Then, the relation $\eta^*$ is strong regular on $R$.*

*Proof* Suppose that $a\eta^*b$ and $y_2, \ldots, y_m$ are arbitrary elements of $R$. It follows that there exist $z_0 = a, z_1 \ldots, z_n = b$ such that for $i \in \{0, 1, \ldots, n-1\}$, we have $z_i \eta z_{i+1}$. From $z_i \eta z_{i+1}$, it follows that there exists $k_i \geq 0$ such that $\{z_i, z_{i+1}\} \subseteq A_1^{k_i m}[x_{ki}]$ and

$$\{f(z_i, y_2^m),\ f(z_{i+1}, y_2^m)\} \subseteq A_1^{(k_i+1)m}[x_{(k_i+1)i}].$$

Hence, every element $u_1 \in f(a, y_2, \ldots, y_m)$ is $\eta$ equivalent to every element $u_2 \in f(b, y_2, \ldots, y_m)$. This completes the proof. □

Let $(R, f)$ be an $m$-ary hypergroup and consider the canonical projection $\pi : R \longrightarrow [R : \eta^*]$. We define $kerl\pi$ by $K(R)$.

**Definition 2.10** Let $(R, f)$ be an $m$-ary hypergroup and $A$ be a non-empty subset of $R$. We say that $A$ is a *complete part* of $R$ if for any non-zero natural number $k$, the following implication holds:

$$A \cap A_1^k[x_{ki}] \neq \emptyset \Longrightarrow A_1^{km}[x_{ki}] \subseteq A.$$

If $A$ is a subset of $R$, we denote by $C(A)$ the *complete closure* of $A$, which is the smallest complete part of $R$, that contains $A$.

Let $(R, f)$ be an $m$-ary hypergroup and $x \in R$. We define

$$[x] = \{A \in \mathcal{P}^*(R) : x \in A, k \in \mathbb{N}, A = A_1^{km}[x_{ki}]\},$$

$$[[x]] = \bigcup_{A \in [x]} A.$$

**Theorem 2.11** *Let $(R, f)$ be an $m$-ary hypergroup. Then, $[[x]]$ is a complete part.*

*Proof* Suppose that $A_1^{km}[x_{ki}] \cap [[x]] \neq \emptyset$ and $a \in A_1^{km}[x_{ki}] \cap [[x]]$. Hence, there exists $a \in [x]$ such that $a \in A_1^{km}[x_{ki}] \cap A$. There exist $z_2, z_3, \ldots, z_m$ and $y_1, y_2, \ldots, y_{m-1}$ such that $x \in f(a, z_2^m)$, $x_{km} \in f(y_1, y_2, \ldots, y_{m-1}, x)$. Then

$$\begin{aligned}
A_1^{km}[x_{ki}] &= f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, x_{km}) \\
&\subseteq f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(y_1^{(m-1)}, x)) \\
&\subseteq f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(y_1^{(m-1)}, f(a, z_2^m))) \\
&\subseteq f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(y_1^{(m-1)}, f(A, z_2^m))),
\end{aligned}$$

and

$$\begin{aligned}
x \in f(a, z_2^m) &\subseteq f(A_1^{km}[x_{ki}], z_2^m) \subseteq f(f(A_1^{(k-1)m}[x_{ki}], x_{k2}^m), z_2^m) \\
&\subseteq f(f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, x_{km}), z_2^m) \\
&\subseteq f(f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(y_1^{m-1}, x)), z_2^m) \\
&\subseteq f(f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(y_1^{m-1}, f(a, z_2^m))), z_2^m) \\
&\subseteq f(f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(y_1^{m-1}, f(A, z_2^m))), z_2^m) \\
&= f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(f(y_1^{m-1}, A), z_2^m)) \\
&= f(A_1^{(k-1)m}[x_{ki}], x_{k2}^{(m-1)}, f(y_1^{m-1}, f(A, z_2^m)).
\end{aligned}$$

This implies that $[[x]]$ is a complete part of $(R, f)$. □

**Proposition 2.12** *For every $x \in K(R)$, $[[x]]] = K(R)$.*

*Proof* We consider the following steps:

(1) Let $A$ be a non-empty set of $R$. We prove

$$f(\underbrace{K(R), K(R), \ldots, K(R)}_{m-1}, A) = \cdots = f(A, \underbrace{K(R), K(R), \ldots, K(R)}_{m-1}) = \pi_R^{-1} \circ \pi_R(A).$$

If $y \in \pi_R^{-1} \circ \pi_R(A)$, then there exists $a \in A$ such that $\pi_R(a) = \pi_R(y)$. On the other hand, there exist $u_2, \ldots, u_m \in R$ such that $y \in f(a, u_2, u_3, \ldots, u_m)$ and $\pi_R(y) = f(\pi_R(a), \pi_R(u_2), \ldots, \pi_R(u_m))$. Hence, $u_i \in K(R)$ for $2 \leq i \leq m$. Therefore, $\pi_R^{-1} \circ \pi_R(A) \subseteq f(a, K(R), \ldots, K(R))$.
Conversely, if $x \in f(a, K(R), \ldots, K(R))$, then $\pi_R(x) = f(\pi_R(a), \pi_R(y_2), \ldots, \pi_R(y_m)))$ where $y_i \in K(R)$ for $2 \leq i \leq m$. Whence $\pi_R(x) = \pi_R(a)$. Therefore, $x \in \pi_R^{-1} \circ \pi_R(A)$. We can see another assertions.

(2) Denote $K_1(A) = A$ and for $n \geq 1$

$$K_{n+1}(A) = \{x \in R : \exists k \in \mathbb{N}^*, x \in A_1^{km}[x_{ki}], K_n(A) \cap A_1^{km}[x_{ki}] \neq \emptyset\}.$$

Let $K(A) = \bigcup_{n\geq 1} K_n(A)$. We prove that $K(A) = C(A)$. Notice that $K(A)$ is a complete part of $R$. Indeed, if we suppose that $K(A) \cap A_1^{km}[x_{ki}] \neq \emptyset$, then there exists $n \geq 1$ such that $K_n(A) \cap A_1^{km}[x_{ki}] \neq \emptyset$ which means that $A_1^{km}[x_{ki}] \subseteq K_{n+1}(A) \subseteq K(A)$.
Now, if $A \subseteq B$ and $B$ is a complete part of $R$, then we prove that $K(A) \subseteq B$.

(3) If $B$ is a non-empty subset of $R$, then $C(B) = \bigcup_{b\in B} C(b)$. Clearly, for $b \in B$, we have $C(b) \subseteq C(B)$. On the other hand, $C(B) = \bigcup_{n\geq 1} K_n(B)$. We shall prove by induction. For $n = 1$, we have $K_1(B) = B = \bigcup_{b\in B} K_1(b)$. Suppose that $K_n(B) \subseteq \bigcup_{b\in B} K_n(b)$. If $z \in K_{n+1}(B)$, then there exists a hyperproduct $A_1^{km}[x_{ki}]$ such that $z \in P$ and $K_n(B) \cap A_1^{km}[x_{ki}] \neq \emptyset$. Then, there exists $b \in B$ such that $K_n(b) \cap A_1^{km}[x_{ki}]$. Hence, $z \in K_{n+1}(b)$. We obtain $K_{n+1}(B) \subseteq \bigcup_{b\in B} K_{n+1}(b)$. Therefore, $C(B) = \bigcup_{b\in B} C(b)$.

(4) For all non-empty subset $A$ of $R$, we prove that

$$f(\underbrace{K(R), \ldots K(R)}_{m-1}, A) = \cdots = f(A, \underbrace{K(R), \ldots K(R)}_{m-1}) = C(A).$$

Indeed, we have $x \in \pi_R^{-1} \circ \pi_R(A)$ if and only if there exists $a \in A$ such that $x\gamma^* a$ which means that $x \in K(a) = C(a)$.

(5) A non-empty subset $A$ is complete if and only if

$$f(K(R), K(R), \ldots, A) = \cdots = f(A, K(R), \ldots, K(R)) = A.$$

(6) We have f(K(R), K(R), …, K(R)) = K(R), which implies that $K(R)$ is a complete part. □

**Theorem 2.13** *Let $R$ be an $m$-ary hypergroup. Then, $\eta^* = \eta$*

*Proof* Suppose that $\eta^*(x) = \eta^*(y)$. Hence, $x \in f(x, w_2, w_3, \ldots, w_m)$ and $y \in f(x, v_2, v_3, \ldots, v_m)$, where $w_2, w_3, \ldots, w_m, v_2, v_3, \ldots, v_m \in K(R)$. On the other hand, $K(R) = [[w_i]]$, for $2 \leq i \leq m$ implies that $A_1^{k_i m}[x_{k_{ij}}]$ exist such that $v_i \in A_1^{k_i m}[x_{k_{ij}}]$. Therefore, $\{w_i, v_i\} \subseteq A_i^{k_i m}[x_{k_{ij}}]$ and for this reason $x\eta y$. □

## 3 Quotient Krasner $(m, n)$-hyperrings

In this section we observe that if $I$ is a normal hyperideal of a Krasner hyperring $R$, then the quotient Krasner $(m, n)$-hyperring $[R : I^*]$ is an $(m, n)$-ring.

A non-empty subset $I$ of a Krasner $(m, n)$-hyperring is an $i$-hyperideal if the following conditions hold:

(i) $(I, f)$ is a canonical $m$-ary hypergroup,
(ii) for every $x_1^n \in R$, $g\left(\overset{(i-1)}{x_1}, I, \overset{n}{x_{i+1}}\right) \in I$.

If for every $1 \leq i \leq n$, $I$ is an i-hyperideal, then $I$ is called a *hyperideal* of $R$.

**Proposition 3.1** [15] *Let $(R, f, g)$ be a Krasner $(m, n)$-hyperring. Then,*

*(i) for every $x \in R$, we have $-(-x) = x$, $-0 = 0$,*

*(ii) for every* $x \in R$, $0 \in f\left(x, -x, \overset{(m-2)}{0}\right)$,
*(iii) for every* $x_1^m$, $-f(x_1, x_2, \ldots, x_m) = f(-x_1, -x_2, \ldots, -x_m)$, *where* $-A = \{-a \mid a \in A\}$.

**Definition 3.2** A hyperideal $I$ of a Krasner $(m, n)$-hyperring $R$ is called *normal* if for every $r \in R$

$$f\left(-r, I, r, \overset{(m-3)}{0}\right) \subseteq I.$$

**Proposition 3.3** *Let* $(R, f, g)$ *be a Krasner* $(m, n)$*-hyperring and* $I$ *be an ideal of* $R$*. Then, a relation*

$$x \equiv y \Longleftrightarrow x \in f\left(I, \overset{(m-2)}{0}, y\right),$$

*is an equivalence relation on* $(R, f, g)$.

*Proof* Suppose that $x \in R$. Since $x = f\left(\overset{(m-1)}{0}, x\right) \in f\left(I, \overset{(m-2)}{0}, x\right)$, then this relation is reflexive. Let $x, y \in (R, f, g)$ and $x \equiv y$. Then, $x \in f\left(I, \overset{(m-2)}{0}, y\right)$. Hence for some $a \in I$, $x \in f\left(a, \overset{(m-2)}{0}, y\right)$. That is

$$y \in f\left(-\overset{(m-2)}{0}, -a, x\right) = f\left(-a, \overset{(m-2)}{0}, x\right) \subseteq f\left(I, \overset{(m-2)}{0}, x\right).$$

So this relation is symmetric. Let $x, y, z \in R$ such that $y \equiv z$ and $y \equiv z$. Then, $x \in f\left(a, \overset{(m-2)}{0}, y\right)$ and $y \in f\left(b, \overset{(m-2)}{0}, z\right)$, for some $a, b \in I$. So

$$\begin{aligned} x \in f\left(a, \overset{(m-2)}{0}, f\left(b, \overset{(m-2)}{0}, z\right)\right) &= f\left(f\left(a, \overset{(m-2)}{0}, b\right), \overset{(m-2)}{0}, z\right) \\ &\subseteq f\left(I, \overset{(m-2)}{0}, z\right). \end{aligned}$$

Therefore, $x \equiv z$ and $\equiv$ is an equivalence relation. □

Let $I^*[x]$ be the equivalence class of the element $x \in R$.

**Lemma 3.4** *Let* $I$ *be an ideal of a Krasner* $(m, n)$*-hyperring. Then,*

$$I^*(x) = f\left(I, \overset{(m-2)}{0}, x\right).$$

*Proof* The proof is straightforward. □

**Proposition 3.5** Let $R$ be a Krasner $(m, n)$-hyperring and $I$ be a normal ideal. Then, for every $x, y \in I$ the following statements are equivalent:

(i) $I^*(x) = I^*(y)$,
(ii) $f\left(x, -y, \overset{(m-2)}{0}\right) \subseteq I$,
(iii) $f\left(x, -y, \overset{(m-2)}{0}\right) \bigcap I \neq \emptyset$.

*Proof* (i) $\Longrightarrow$ (ii) There exists $a \in I$ such that $y \in f\left(a, \overset{(m-2)}{0}, x\right)$. Then, we have

$$
\begin{aligned}
f\left(x, -y, \overset{(m-2)}{0}\right) &\subseteq f\left(x, f\left(-a, \overset{(m-2)}{0}, -x\right) \overset{(m-2)}{0}\right)\\
&= f\left(x, f\left(-x, -a, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\\
&\subseteq f\left(f\left(x, I, -x, \overset{(m-3)}{0}\right), \overset{(m-1)}{0}\right) \subseteq f\left(I, \overset{(m-1)}{0}\right) \subseteq I.
\end{aligned}
$$

(ii)$\Longrightarrow$ (iii) Obviously. (iii)$\Longrightarrow$ (i) Since $f\left(x, -y, \overset{(m-2)}{0}\right) \bigcap I \neq \emptyset$, there exists $a \in I$ such that $a \in f\left(x, -y, \overset{(m-2)}{0}\right)$. This implies that $x \in f\left(a, \overset{(m-2)}{0}, y\right)$. Hence

$$
\begin{aligned}
f\left(-y, x, \overset{(m-2)}{0}\right) &\subseteq f\left(-y, f\left(a, \overset{(m-2)}{0}, y\right), \overset{(m-2)}{0}\right)\\
&= f\left(f\left(-y, a, \overset{(m-3)}{0}\right), \overset{(m-1)}{0}\right) \subseteq I.
\end{aligned}
$$

Then, $z \in f\left(-y, x, \overset{(m-2)}{0}\right)$ implies that $z \in I$. Hence $-y \in f\left(z, -x, \overset{(m-2)}{0}\right)$. Therefore, $y \in f\left(-z, x, \overset{(m-2)}{0}\right) \subseteq f\left(I, \overset{(m-1)}{0}, x\right)$. This completes the proof. $\square$

We note that when $I$ is a normal hyperideal, $x \equiv y$ if and only if

$$
f\left(x, -y, \overset{(m-2)}{0}\right) \subseteq I.
$$

**Theorem 3.6** *Let $(R, f, g)$ be a Krasner $(m, n)$-hyperring. Then, the following sets are equal:*

$$
\Omega_1 = \left\{I^*(z) : z \in f\left(x, y, \overset{(m-2)}{0}\right)\right\}, \quad \Omega_2 = \left\{I^*(z) : z \in f\left(I^*(x), I^*(y), \overset{(m-2)}{0}\right)\right\},
$$
$$
\Omega_3 = \left\{I^*(z) : I^*(z) \subseteq f\left(I^*(x), I^*(y), \overset{(m-2)}{0}\right)\right\}.
$$

*Proof* Suppose that $I^*(z) \in \Omega_1$. Then $z \in f\left(x, y, \overset{(m-2)}{0}\right)$. This implies that $\Omega_1 \subseteq \Omega_2$. Let $I^*(z) \in \Omega_2$. Then, $z \in f\left(I^*(x), I^*(y), \overset{(m-2)}{0}\right)$. Hence $z \in f\left(f\left(I, \overset{(m-2)}{0}, x\right), f\left(I, \overset{(m-2)}{0}, y\right), \overset{(m-2)}{0}\right)$, implies that

$$
z \in f\left(I, f\left(\overset{(m-2)}{0}, x, f\left(I, \overset{(m-2)}{0}, y\right)\right), \overset{(m-2)}{0}\right)
$$

$$
= f\left(I, f\left(f\left(x, y, \overset{(m-2)}{0}\right), I, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right).
$$

Hence $z \in f\left(I, f\left(z_1, I, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)$ for some $z_1 \in f\left(x, y, \overset{(m-2)}{0}\right)$. Therefore,

$$
z \in f\left(f\left(I, I, \overset{(m-2)}{0}\right), z_1, \overset{(m-2)}{0}\right) \subseteq f\left(I, z_1, \overset{(m-2)}{0}\right) = I^*(z_1).
$$

Thus, $I^*(z) = I^*(z_1)$ where $z_1 \in f\left(x, y, \overset{(m-2)}{0}\right)$. Hence $\Omega_2 \subseteq \Omega_1$.

Suppose that $I^*(z) \in \Omega_1$. Therefore,

$$\begin{aligned} f\left(I^*(x), I^*(x), \overset{(m-2)}{0}\right) &= f\left(f\left(I, \overset{(m-2)}{0}, x\right), f\left(I, \overset{(m-2)}{0}, y\right), \overset{(m-2)}{0}\right) \\ &= f\left(f\left(f\left(I, I, \overset{(m-2)}{0}\right), x, \overset{(m-2)}{0}\right), y, \overset{(m-2)}{0}\right) \\ &= f\left(f\left(I, x, \overset{(m-2)}{0}\right), y, \overset{(m-2)}{0}\right) \\ &= f\left(f\left(x, y, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}, I\right). \end{aligned}$$

This implies that $I^*(z) \subseteq f\left(I^*(x), I^*(y), \overset{(m-2)}{0}\right)$. Thus, $\Omega_1 \subseteq \Omega_3$. This completes the proof. □

**Theorem 3.7** *Let $R$ be a Krasner $(m, n)$-hyperring and $I$ be a hyperideal of $R$. Then, $[R : I^*] = \{I^*(x) : x \in R\}$ is a Krasner $(m, n)$-hyperring with respect to the following $m$-ary hyperoperation:*

$$\begin{aligned} &f/I(I^*(x_1), I^*(x_2), \ldots, I^*(x_m)) = \{I^*(z) : z \in f(x_1^m)\}, \\ &g/I(I^*(x_1), I^*(x_2), \ldots, I^*(x_m)) = g(x_i^n) \end{aligned}$$

*Proof* Omitted as obvious. □

**Definition 3.8** Let $R$ be a Krasner $(m, n)$-hyperring and $R_1$ be a subcanonical $m$-ary hypergroup of $R$. We denote

$$\Omega^e(R_1, R) = \left\{x \in R : f\left(\overset{(m/2)-1}{x_1}, -\overset{(m)}{x_{(m/2)}}\right) \subseteq R_1\right\},$$

when $m$ is even and

$$\Omega^o(R_1, R) = \left\{x \in R : f\left(\overset{(m-1/2)-1}{x_1}, -\overset{(m-1)}{x_{(m-1)/2}}, 0\right) \subseteq R_1\right\},$$

when $m$ is odd, where for every $1 \le i \le n$, $x_i = x$.

**Proposition 3.9** *Let $(R, f, g)$ be a Krasner $(m, n)$-hyperring and $R_1$ be a sub-canonical $m$-hypergroup of $(R, f)$. Then,*

$$\Delta(R_1, R) = \left\{x \in R : f\left(x, -x, \overset{(m-2)}{0}\right) \subseteq R_1\right\},$$

*is a sub-canonical hypergroup of $(R, f)$ containing $R_1$.*

*Proof* Suppose that $x \in R_1$, then $f\left(x, -x, \overset{(m-2)}{0}\right) \subseteq R_1$. Indeed, $R_1$ is a sub-canonical hypergroup. Then, $R_1 \subseteq \Delta(R_1, R)$. Let $x_1, x_2, \cdots x_m \in \Delta(R_1, R)$. Then, for every $z \in f(x_1, x_2, \cdots, x_m)$, we get

$$\begin{aligned} f\left(z, -z, \overset{(m-2)}{0}\right) &\subseteq f\left(f(x_1, x_2, \ldots, x_m), -f(x_1, x_2, \ldots, x_m), \overset{(m-2)}{0}\right) \\ &= f\left(f(x_1, x_2, \ldots, x_m), f(-x_1, -x_2, \ldots, -x_m), \overset{(m-2)}{0}\right) \\ &= f\left(f(x_1, x_2, \ldots, x_m), f(-x_1, -x_2, \ldots, -x_m), \underbrace{f\left(\overset{(m)}{0}\right), \ldots, f\left(\overset{(m)}{0}\right)}_{m-2}\right) \\ &= f\left(f\left(x_1, -x_1, \overset{(m-2)}{0}\right), \ldots, f\left(x_m, -x_m, \overset{(m-2)}{0}\right)\right) \\ &\subseteq f\,(R_1, R_1, \ldots, R_1) \subseteq R_1. \end{aligned}$$

This completes the proof. □

**Proposition 3.10** Let $R$ be a Krasner $(m, n)$-hyperring. Then, for every $x, y \in \Delta(0, R)$, $f\left(x, y, \overset{(m-2)}{0}\right)$ is singleton.

*Proof* Suppose that $x, y \in \Delta(0, R)$. Consider the set $f\left(x, y, \overset{(m-2)}{0}\right)$ and $a, b \in f\left(x, y, \overset{(m-2)}{0}\right)$. Then we have

$$\begin{aligned} f\left(a, -b, \overset{(m-2)}{0}\right) &\in f\left(f\left(x, y, \overset{(m-2)}{0}\right), f\left(-x, -y, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\\ &= f\left(x, f\left(y, \overset{(m-2)}{0}, f\left(-x, -y, \overset{(m-2)}{0}\right)\right), \overset{(m-2)}{0}\right)\\ &= f\left(x, f\left(-x, \overset{(m-1)}{0}\right), \overset{(m-2)}{0}\right) = 0. \end{aligned}$$

This implies that $f\left(a, -b, \overset{(m-2)}{0}\right) = 0$. Hence

$$\begin{aligned} a = f\left(a, \overset{(m-1)}{0}\right) &\in f\left(f\left(b, -b, \overset{(m-2)}{0}\right), a, \overset{(m-2)}{0}\right)\\ &= f\left(b, f\left(a, -b, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\\ &= f\left(b, \overset{(m-1)}{0}\right) = b. \end{aligned}$$

Therefore, $f\left(x, y, \overset{(m-2)}{0}\right)$ is a singleton. □

**Proposition 3.11** *Let $R$ be a Krasner $(m, n)$-hyperring such that $m$ is even and $R_1$ be a sub-canonical $m$-hypergroup of $(R, f)$. Then, $\Omega^e(R_1, R)$ is a sub-canonical $m$-ary hypergroup of $(R, f)$, where $x_i = x$, for $1 \leq i \leq n$.*

*Proof* Suppose that $x_1, x_2, \ldots, x_m \in \Omega^e(R_1, R)$. Then

$$f\left(\overset{(m/2)-1}{x_{11}}, -\overset{m}{x_{1(m/2)}}\right), \ldots, f\left(\overset{(m/2)-1}{x_{m1}}, -\overset{m}{x_{m(m/2)}}\right) \subseteq R_1.$$

Let $r \in f(x_1, x_2, \ldots, x_m)$. Then, we have

$$\begin{aligned} f\left(\overset{(m/2)-1}{r_1}, -\overset{m}{r_{(m/2)}}\right) &\subseteq f(f(x_1, x_2, \ldots, x_m), \ldots, f(-x_1, -x_2, \ldots, -x_m))\\ &= f\left(f\left(\overset{(m/2)-1}{x_{11}}, -\overset{m}{x_{1(m/2)}}\right), \ldots, f\left(\overset{(m/2)-1}{x_{m1}}, -\overset{m}{x_{m(m/2)}}\right)\right)\\ &\subseteq f(R_1, \ldots, R_1)\\ &\subseteq R_1. \end{aligned}$$

This implies that $\Omega^e(R_1, R)$ is a sub-canonical $m$-ary hypergroup of $(R, f)$. □

**Proposition 3.12** *Let $R$ be a Krasner $(m, n)$-hyperring such that $m$ is even. Then, $\Omega^e(0, R)$ is an $m$-ary group of $(R, f)$ containing all $m$-ary subgroups of $R$.*

*Proof* By Proposition 3.11, $\Omega^e(0, R)$ is a sub-canonical $m$-ary hypergroup of $R$. Let $x_1, x_2, \ldots, x_m \in \Omega^e(0, R)$ and $a, b \in f(x_1, \ldots, x_m)$. Then, we have

$$\begin{aligned} f\left(\overset{(m/2)-1}{a_1}, -\overset{(m)}{b_{(m/2)}}\right) &\subseteq f(f(x_1, \ldots, x_m), \ldots, -f(x_1, \ldots, x_m))\\ &= f\left(f\left(\overset{(m/2)-1}{x_{11}}, -\overset{m}{x_{(m/2)}}\right), \ldots, f\left(\overset{(m/2)-1}{x_{m1}}, -\overset{(m)}{x_{(m/2)}}\right)\right)\\ &= \{0\}. \end{aligned}$$

This implies that $f\left(\overset{(m/2)-1}{a_1}, -\overset{(m)}{b_{(m/2)}}\right) = \{0\}$. Therefore,

$$\begin{aligned} f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-1)}{b_{(m/2)}}, 0\right) &\subseteq f\left(f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-1)}{b_{(m/2)}}, 0\right), \overset{(m-1)}{0}\right) \\ &\subseteq f\left(f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-1)}{b_{(m/2)}}, 0\right), f\left(b, -b, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right) \\ &= f\left(f\left(f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-1)}{b_{(m/2)}}, 0\right), -b, \overset{(m-2)}{0}\right), b, \overset{(m-2)}{0}\right) \\ &= f\left(f\left(f\left(\overset{(m/2)-1}{a_1}, -\overset{m}{b_{(m/2)}}\right), \overset{(m-1)}{0}\right), b, \overset{(m-2)}{0}\right). \end{aligned}$$

This implies that $f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-1)}{b_{(m/2)}}, 0\right) = \{0\}$. By continuing this process $f\left(a, -b, \overset{(m-2)}{0}\right) = \{0\}$. We have $f\left(b, \overset{(m-1)}{0}\right) \subseteq f\left(f\left(a, -b, \overset{(m-2)}{0}\right), b, \overset{(m-2)}{0}\right)$. Thus,

$$\begin{aligned} a = f\left(a, \overset{(m-2)}{0}\right) &\subseteq f\left(a, f\left(b, -b, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right) \\ &= f\left(f\left(a, -b, \overset{(m-2)}{0}\right), b, \overset{(m-2)}{0}\right) \\ &= f\left(b, \overset{(m-1)}{0}\right) = b. \end{aligned}$$

Hence, $a = b$. This means that $f(x_1, x_2, \ldots, x_m)$ has only one element. Therefore, $\Omega^e(R_1, R)$ is an $m$-ary hypergroup. Suppose that $R_1$ is any $m$-ary subgroup of $R$. Then for every $x \in R_1$, $-x \in R_1$ and $f\left(\overset{(m/2)-1}{x_1}, -\overset{m}{x_{(m/2)}}\right) = \{0\}$. Hence $R_1 \subseteq \Omega^e(R_1, R)$. □

**Proposition 3.13** *Let $R$ be a Krasner $(m, n)$-hyperring. Then, $\Omega^e(0, R) = R$ if and only if $\{0\}$ is a normal ideal of $R$.*

*Proof* Suppose that $\Omega^e(0, R) = R$. By Proposition 3.12, $\Omega^e(0, R)$ is an $m$-ary group of $(R, f)$. We show that $\{0\}$ is a normal ideal. We have $f\left(\overset{(m/2)-1}{x_1}, -\overset{m}{x_{(m/2)}}\right) = \{0\}$, for every $x \in R$. This implies that $f\left(x, -x, \overset{(m-2)}{0}\right) = \{0\}$. Hence $f\left(x, 0, -x, \overset{(m-3)}{0}\right) = \{0\}$. This implies that $\{0\}$ is normal.

Conversely, assume that $x \in R$. Since $\{0\}$ is normal,

$$f\left(x, -x, \overset{(m-2)}{0}\right) = \{0\}.$$

We prove that $f\left(\overset{(m/2)-1}{x_1}, -\overset{(m)}{x_{(m/2)}}\right) = \{0\}$. We have

$$\begin{aligned} &f\left(\overset{(m/2)-1}{x_1}, -\overset{(m)}{x_{(m/2)}}\right) \subseteq f\left(f\left(\overset{(m/2)-1}{x_1}, -\overset{(m)}{x_{(m/2)}}\right), \overset{(m-1)}{0}\right) \\ &= f\left(f\left(x, -x, \overset{(m-2)}{0}\right), \overset{(m/2)-1}{x_1}, \overset{(m-1)}{x_{(m/2)-1}}, 0\right) \\ &= f\left(\overset{(m/2)-2}{x_1}, -\overset{(m-1)}{x_{(m/2)-1}}, \overset{(2)}{0}\right) \end{aligned}$$

$$= f\left(\overset{(m/2)-2}{x_1}, -\overset{(m-1)}{x_{(m/2)-1}}, f\left(\overset{(m)}{0}\right), 0\right)$$
$$= f\left(f\left(x, -x, \overset{m-2}{0}\right), \overset{(m/2)-3}{x_1}, -\overset{(m-1)}{x_{(m/2)-2}}, \overset{(3)}{0}\right)$$
$$= f\left(\overset{(m/2)-3}{x_1}, -\overset{(m-1)}{x_{(m/2)-2}}, \overset{(4)}{0}\right).$$

By continuing this process, we have

$$f\left(\overset{(m/2)-1}{x_1}, -\overset{(m)}{x_{(m/2)}}\right) = f\left(x, -x, \overset{(m-2)}{0}\right) = \{0\}.$$

This implies that $x \in \Omega^e(0, R)$. Therefore, $\Omega^e(0, R) = R$. □

**Corollary 3.14** *Let $I$ be a hyperideal of a Krasner $(m, n)$-hyperring such that $m$ is even. Then, $I$ is normal if and only if $\Omega^e(I, R) = R$*

**Proposition 3.15** *Let $R$ be a Krasner $(m, n)$-hyperring, and $R_1$ be a sub-canonical $m$-ary hypergroup of $(R, f)$. Then,*

$$\Omega^o(R_1, R) = \left\{x \in R : f\left(\overset{(m/2)-1}{x_1}, -\overset{(m-1)}{x_{(m/2)}}, 0\right) \subseteq R_1\right\},$$

*is a sub-canonical $m$-ary hypergroup of $(R, f)$.*

*Proof* Suppose that $x_1, x_2, \ldots, x_m \in \Omega^o(R_1, R)$ and $r \in f(x_1, x_2, \ldots, x_m)$. Then we get

$$f\left(\overset{(m/2)-1}{r_1}, -\overset{(m-1)}{r_{(m/2)}}, 0\right) \subseteq f(f(x_1, x_2, \ldots, x_m), \ldots, -f(x_1, x_2, \ldots, x_m), f(0, \ldots, 0))$$

$$= f\left(f\left(\overset{(m/2)-1}{x_{11}}, -\overset{(m-1)}{x_{1(m/2)}}, 0\right), \ldots, f\left(\overset{(m/2)-1}{x_{m1}}, -\overset{(m-1)}{x_{m(m/2)}}, 0)\right)\right)$$

$$\subseteq f(R_1, R_1, \ldots, R_1) \subseteq R_1.$$

Therefore, $f(x_1, x_2, \ldots, x_m) \subseteq R_1$. This completes the proof. □

**Proposition 3.16** *Let $R$ be a Krasner $(m, n)$-hyperring. Then, $\Omega^o(0, R)$ is an $m$-ary subgroup of $(R, f)$ containing all subgroups of $R$.*

*Proof* By Proposition 3.15, $\Omega^o(0, R)$ is the sub-canonical $m$-ary hypergroup of $(R, f)$. Let $x_1, x_2, \ldots, x_m \in \Omega^o(0, R)$ and $a, b \in f(x_1, x_2, \ldots, x_m)$. Then,

$$f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-1)}{b_{(m/2)}}, 0\right) \subseteq f\left(f(x_1, x_2, \ldots, x_m), \ldots, -f(x_1, x_2, \ldots, x_m), f\left(\overset{(m)}{0}\right)\right)$$
$$= f\left(f\left(\overset{(m/2)-1}{x_{11}}, -\overset{(m-1)}{x_{1(m/2)}}, 0\right), \ldots, f\left(\overset{(m/2)-1}{x_{11}}, -\overset{(m-1)}{x_{1(m/2)}}, 0\right)\right)$$
$$= f\left(\overset{(m)}{0}\right) = 0.$$

This implies that

$$f\left(\overset{(m/2)-1}{a_1}, -\overset{m-2}{b_{(m/2)}}, \overset{(2)}{0}\right) \subseteq f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-2)}{b_{(m/2)}}, f\left(b, -b, \overset{(m-2)}{0}\right), 0\right)$$
$$= f\left(f\left(\overset{(m/2)-1}{a_1}, -\overset{(m-1)}{b_{(m/2)}}, 0\right), b, \overset{(m-2)}{0}\right).$$

By continuing this process $f\left(a, -b, \overset{(m-2)}{0}\right) = \{0\}$. Therefore, $a = b$. □

**Proposition 3.17** *Let $R$ be a Krasner $(m,n)$-hyperring. Then, $R$ is an $(m,n)$-ring if and only if $\{0\}$ is a normal ideal of $R$.*

*Proof* By Proposition 3.12, $R$ is an $(m,n)$-ring if and only if $\Omega^e(0,R) = R$ and by Proposition 3.13, $\Omega^e(0,R) = R$ if and only if $\{0\}$ is a normal ideal of $R$. □

**Proposition 3.18** *Let $R$ be a Krasner $(m,n)$-hyperring. Then, $\Omega^0(0,R) = R$ if and only if $\{0\}$ is a normal hyperideal of $R$.*

*Proof* The proof is similar to 3.17. □

*Remark 3.19* If $R$ is a Krasner $(m,n)$-hyperring and $I$ is a hyperideal of $R$, then $[R : I^*]$ is also a Krasner $(m,n)$-hyperring. Moreover, if $I$ is a normal hyperideal of $R$, then by Theorems 3.17 and 3.18, $[R : I^*]$ is an $(m,n)$-ring. Hence, the quotient Krasner $(m,n)$-hyperrings considered in [15] are just $(m,n)$-rings. So in the isomorphism theorems proved in [15], all quotient hyperrings considered are $(m,n)$-rings.

**Theorem 3.20** *Let $(R, f, g)$ be a multiplicative $(m,n)$-ary hyperring. Then, the following statements are equivalent:*

*(i) there exists $a \in R$ such that $\left|g\left(a_{\sigma(1)}, 0_{\sigma(2)}, 0_{\sigma(3)}, \ldots, 0_{\sigma(n)}\right)\right| = 1$, for every $\sigma \in S_n$,*
*(ii) $|g(0_{\sigma(1)}, 0_{\sigma(2)}, \ldots, 0_{\sigma}(n))| = 1$,*
*(iii) for all $a_1, a_2, \ldots, a_n \in R$, $|g(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(3)}| = 1$,*
*(iv) $(R, f, g)$ is an $(m,n)$-ring.*

*Proof* $(i) \Longrightarrow (ii)$. Suppose that $0 \neq a \in R$ and $|g(a_{\sigma(1)}, 0_{\sigma(2)}, \ldots, 0_{\sigma}(n))| = 1$ for every $\sigma \in S_n$. We have

$$\begin{aligned} g\left(0_{\sigma(1)}, \overset{\sigma(n)}{0_{\sigma(2)}}\right) &= g\left(f\left(a_{\sigma(1)}, -a_{\sigma(2)}, \overset{(m-2)}{0}\right), \ldots, 0_{\sigma(m)}\right) \\ &= f\left(g\left(a_{\sigma(1)}, \overset{(n-1)}{0}\right), g\left(-a_{\sigma(2)}, \overset{(n-1)}{0}\right), \overset{(m-2)}{0}\right) \\ &= f\left(g\left(a_{\sigma(1)}, \overset{(n-1)}{0}\right), -g\left(a_{\sigma(2)}, \overset{(n-1)}{0}\right), \overset{(m-2)}{0}\right), \qquad * \end{aligned}$$

this implies that $|g(0_{\sigma(1)}, 0_{\sigma(2)}, \ldots, 0_{\sigma}(n))| = 1$.

$(ii) \Longrightarrow (iii)$. Let $a \in R$. Then, by $*$ and $(ii)$, we have $\left|g\left(a_{\sigma(1)}, \overset{(n-1)}{0}\right)\right| = 1$. Indeed, if we suppose that $x \neq y$ are elements of $g\left(a_{\sigma(1)}, \overset{(n-1)}{0}\right)$, then $g\left(\overset{\sigma(n)}{0_{\sigma(1)}}\right)$ would contain $f\left(x, -y, \overset{(m-2)}{0}\right) \neq 0$, that is a contradiction. Since,

$$\begin{aligned} g\left(a_{\sigma(1)}, \overset{\sigma(n)}{0_{\sigma(2)}}\right) &= g\left(a_{\sigma(1)}, f\left(a_{\sigma(2)}, -a_{\sigma(2)}, \overset{(m-2)}{0}\right), \ldots, 0_{\sigma(m)}\right) \\ &= f\left(g\left(a_{\sigma(1)}, a_{\sigma(2)}, \overset{(n-3)}{0}\right), g\left(-a_{\sigma(2)}, \overset{(n-2)}{0}\right), \overset{(m-2)}{0}\right) \\ &= f\left(g\left(a_{\sigma(1)}, a_{\sigma(2)}, \overset{(n-3)}{0}\right), -g\left(a_{\sigma(2)}, \overset{(n-1)}{0}\right), \overset{(m-2)}{0}\right), \qquad * \end{aligned}$$

This implies that $\left|g\left(a_{\sigma(1)}, a_{\sigma(2)}, \overset{(n-3)}{0}\right)\right| = 1$. By continuing this processes $\left|g\left(\overset{\sigma(m)}{a_{\sigma(1)}}\right)\right| = 1$. The other implications $(iii) \Longrightarrow (iv)$ and $(iv) \Longrightarrow (i)$ are immediate. □

**Proposition 3.21** *Let $(R, f, g)$ be a multiplicative $(m,n)$-ary hyperring and $I$ be a hyperideal. Then, $[R : I^*]$ is multiplicative with respect to the following operations: $m$-ary operations and $n$-ary hyperoperations:*

$$\overline{f}\left(I^*[x_1], I^*[x_2], \ldots, I^*[x_m]\right) = I^*\left[f\left(x_1^m\right)\right],$$

$$\overline{g}\left(I^*[x_1], I^*[x_2], \ldots, I^*[x_n]\right) = \left\{I^*[c] : c \in g\left(x_1^{(n)}\right)\right\}.$$

*Proof* The proof is straightforward. □

**Proposition 3.22** *Let $R$ be a multiplicative $(m, n)$-ary hyperring and $I$ be a normal hyperideal of $R$. Then, $[R : I^*]$ is an $(m, n)$-ring.*

*Proof* Suppose that $I^*[x]$ is an element of $[R : I^*]$, then we have $\overline{g}(I^*(x), I, \ldots, I) = \{I^*(c) : c \in g(x, \underbrace{0, \ldots, 0}_{n-1})\}$. Since $0 \in I$, it follows that $g(x, \underbrace{0, \ldots, 0}_{n-1}) \subseteq I$. Hence, $\overline{g}(I^*(x), I, \ldots, I)$ contains only the zero element of $[R : I^*]$. This completes the proof. □



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